MODULE-I: PROBABILITY THEORY AND RANDOM PROCESSES

Basic elements of probability theory:

Probability theory can be approached either using the oretical mathematics or through empirical reasoning. The mathematical approach embed probability theory with in a study of abstract set theory. In contrast, the empirical approach satisfies one's intuition. In our basic study of communication, we will find the empirical approach to be sufficient, though advanced study and referring to current literature will require extending these concepts using principle of set theory.

Some important terms related to probability theory are defined below-

An experiment is a set of rules governing an operation which is performed.

An outcome is the result realized after performing the experiment one time.

An event is combination of outcomes.

Probability :

Let an experiment is performed N times , where N is very large. Also, suppose that in n of these N experiments, the outcome belongs to an event A.

If N is large enough, the probability of event A is given by the ratio n/N.

Formally, we define the probability of an event A as $P(A)=\lim_{N\to\infty} \frac{n_A}{N}$

Where $n_A$is the number of times that the event "A" occurs in N performance of the experiment. From the above equation it is follows that $0\leq P(A)\leq 1$. Here P(A)=0 means the event is impossible and P(A)=1 means the event is sure event.

Consider two different events, A and B, with probabilities

$P(A)=\lim_{N\to\infty} \frac{n_A}{N}$ and $P(B)=\lim_{N\to\infty} \frac{n_B}{N}$

If A and B could not possibly occur at the same time , we call them **disjoint or mutually exclusive**.

The probability of event A or event B is the ratio of the number of times A or B occurs divided by N. If A and B are disjoint, then this is seen to be P(A or B)= P(A U B)$=\lim_{N\to\infty} \frac{n_A+n_B}{N}$ = P(A) + P(B)

Conditional Probabilities:

The probability of event A given that event B has occurred is defined as

$P(A/B) = \frac{P(A \text{ AND } B)}{P(B)} = \frac{P(AB)}{P(B)}$ provided P(B) ≠0 ……..(1)

where the probability of occurrence of the joint event AB is given by P(AB) = P(A) P(B/A)…….(2)

usingeq(1) & (2) $P(A/B)= \frac{P(A)\,P(B/A)}{P(B}$…….(3)

Similarly, the probability of event B given that event A has occurred is defined as

$P(B/A)= \frac{P(B)\,P(A/B)}{P(A)}$…….(4)

Equations (3) & (4) are called **Bayes' rule** where one conditional probability id expressed in term of reversed conditional probability.

Example:A random experiment consist of drawing two cards from a deck in succession (with out replacing the first card drawn). Determine the probability of obtaining two red aces in two draws.

Solution: A= event "red ace in the first draw"

B= event "red ace in the second draw"

We wish to determine P(AB) =P(A) P(B/A)

Here P(A)=2/52=1/26 and P(B/A) is the probability of drawing a red ace in second draw given that the first draw is a red ace.

P(B/A)=1/51

Hence P(AB)=(1/26) (1/51)=1/1326 (Ans.)

Independent Events:

If two or more events are entirely independent that is, the occurrence of one event in no way influences the occurrence of other events then the events are called independent events.

Let A and B are two independent event then,

i.P(B/A)=P(B)

ii.P(A/B)=P(A)

iii. P(AB)= P(A) P(B)

Random Variable(RV):

The outcome of a random experiment may be a real number (as in the case of rolling a die) or it may be non-numerical and described by a phase (such as "heads" and "tails" in tossing a coin). From mathematical point of view, it is desirable to have numerical values for all outcomes. For this reason, we assign a real number to each sample point according to some rule. If there are m sample points $\zeta_1, \zeta_2, \ldots\ldots \zeta_m$, then using some conventional rule, we assign a real number $x(\zeta_i)$ for sample point ζ_i, i=1, 2,….m. Thus X(.) is a function that maps sample points $\zeta_1, \zeta_2, \ldots\ldots \zeta_m$into real numbers $x1, x2, \ldots xn$. Here m is not necessarily equal to n as more than one sample point can map into one value of X.

X is known as the random variable that takes on values $x1, x2, \ldots xn$. We use capital letters(for example X) to denote a random variable and lowercase letters(for example $x1, x2, \ldots xn$) to denote the values they can take on. The probability of an random variable (RV) X taking a value xi is denoted by $P_X(x\text{i})$.

Discrete Random Variable:

A discrete random variable may be defined as the random variable which can take on only finite number of values in a finite observation interval.

A random variable is discrete if there exist a denumerable sequence of distinct numbers xi such that

$$\sum_i P_X(xi) = 1$$

Thus discrete RV can only assume certain discrete values.

Let us consider two RVs X & Y where X can take values $x1, x2, \ldots xn$ and Y can take values from $y1, y2, \ldots ym$. The joint probability that $X=xi$ and $Y=yj$ is denoted by $P_{XY}(xi,yj)$. In this case $\sum_i \sum_j P_{XY}(xi, yj) = 1$

Conditional Probabilities:

If X and Y are two RVs, then the conditional probability of $X=xi$ given $Y=yj$ is denoted by $P_{X|Y}(xi/yj)$.

Moreover $\sum_i P_{X|Y}(xi|yj) = \sum_j P_{Y|X}(yj|xi) = 1$

Using Bayes' rule

$P_{XY}(xi,yj) = P_{X/Y}(xi/yj)\, P_Y(yj) = P_{Y/X}(yj/xi)\, P_X(xi)$

So, $\sum_i P_{XY}(xi, yj) = \sum_i P_{X|Y}(xi|yj) P_Y(yj) = P_Y(yj) \sum_i P_{X|Y}(xi|yj) = P_Y(yj)$

Similarly, $\sum_j P_{XY}(xi, yj) = P_X(xi)$

The probabilities $P_X(xi)$ and $P_Y(yj)$ are known **as marginal profanities**.

Cumulative Distribution Function (CDF):

The cumulative distribution function (CDF) $F_X(x)$ of an random variable (RV) X is the probability that X takes a value less than or equal to x; that is $F_X(x) = P(X \leq x)$.

A CDF $F_X(x)$ has the following properties:

i) $F_X(x) \geq 0$
ii) $F_X(\infty) = 1$
iii) $F_X(-\infty) = 0$
iv) $F_X(x)$ is a non decreasing function i.e $F_X(x1) \leq F_X(x2)$ for $x1 \leq x2$.

Continuous Random Variables:

An RV that can assume any value from a continuous interval is called a continuous RV. In a continuum of any range, an unccountably infinite number of possible values exist, and $P_X(xi)$, the probability that $X=xi$, is one of the uncountably infinite values and is generally zero.

Properties of CDF discussed earlier are general and are valid for continuous as well as discrete RVs.

Probability Density Function (PDF):

The derivative of cumulative distribution function (CDF) $F_X(x)$ with respect to x is known as the probability density function (PDF) and denoted by $f_X(x)$.

$f_X(x) = \frac{d}{dx} F_X(x)$

PDF is the more convenient representation for continuous RVs.

A PDF $f_X(x)$ has the following properties:

i. $f_X(x) \geq 0$ for all values of x.
ii. $\int_{-\infty}^{\infty} f_X(x)dx = 1$
iii. $\int_{-\infty}^{\infty} f(x)dx = F_X(x)$
iv. $P(x1 < X \leq x2) = \int_{x1}^{x2} f_X(x)dx$

Joint Cumulative Distribution function:

For two RVs X and Y we define the joint CDF, $F_{XY}(x,y)$ as the probability that $X \leq x$ and $Y \leq y$.

Thus, $F_{XY}(x,y)=P(X \leq x, Y \leq y)$.

The joint CDF has the following properties:

i. $F_{XY}(x,y) \geq 0$
ii. The joint CDF is monotone non-decreasing function of both *x* and *y*.
iii. The joint CDF is always continuous everywhere in the xy-plane.

Joint Probability Density Function:

The joint PDF is the PDF of two or more RVs. The joint PDF of any two RVs X and Y may be defined as the partial derivative of the joint CDF$F_{XY}(x,y)$ with respect to *x* and *y*.

Mathematically, $f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$

The joint CDF has the following properties:

i. $f_{XY}(x,y) \geq 0$
ii. The total volume under the surface of joint PDF is equals to unity.
Mathematically, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)dx\,dy = 1$
iii. The joint PDF is continuous everywhere because joint CDF is continuous.

Conditional Probability Density Function:

Out of the two random variables, one variable may take a fixed value. In this case, the PDF is called conditional PDF. Out of two random variables, X and Y, let X=*x*. then we may find the conditional PDF of Y given that X=*x* as,

$f_Y(y/x)=\frac{f_{XY}(x,y)}{f_{X(x)}}$

similarly, the conditional PDF of X given that Y=y i.e$f_X(x/y)=\frac{f_{XY}(x,y)}{f_{Y(y)}}$

The conditional PDF has the following properties:

i. $f_X(x/y) \geq 0$ and $f_Y(y/x) \geq 0$
ii. The area under the conditional PDF is equals to unity. Mathematically,

$\int_{-\infty}^{\infty} f_X(x|y)dx = 1and\int_{-\infty}^{\infty} f_Y(y|x)dy = 1$

iii. If two RVs X and Y are statistically independent, then
$f_Y(y/x) = f_Y(y)$and $f_X(x/y) = = f_X(x)$

Statistical Averagesof Random Variables:

The PDF gives some kind of information about the RV but the interpretation of this information quite complex. There are some other measures or numbers which give more useful and quick information about the RV. Collectively these measures or numbers are known as statistical averages. Some useful and convenient informations like mean or average, moments, standard deviation, variance etc. may be obtained from RV. these factors are special characteristics of the PDF and actually they depends on the particular type of PDF.

Mean or Average:

The mean or average of any random variable is expressed by the summation of the values of RV X weighted by their probabilities. Mean value of a RV is denoted by m_x. Mean value is also known as the expected value of a RV X.

i.e m_x=E[X], where E[ ] represents expectation operator.

In general term, mean or average is given by ,

Mean or average of X = $\frac{arithmatic\ \ sum\ of\ all\ values\ of\ X}{totalnumber\ \ of\ values\ of\ X}$

Mean Value of Discrete Random Variable:

Let the discrete RV X takes the following values

X={ *x1, x2, ……..,xn*}. Then the mean or average value is expressed as

m_x=E[X] = $\bar{X}$=*x1* P(*x1*) +*x2* P(*x2*) + ……+*xn* P(*xn*) = $\sum_{i=1}^{n} xi\ P(xi)$

Mean Value of Continuous Random Variable:

For continuous random variable X, the mean or average value is expressed as -

m_x= E[X] = $\bar{X}$ = $\int_{-\infty}^{\infty} x\ f_X(x)\ dx$

Mean Value of a Function of Random Variable :

It is often necessary to find the mean value of a function of RV. We may seek the mean value of a RV Y that is a function of the RV X; that is, we wish to find $\bar{Y}$ where Y= g(X).

Let X be a **discrete RV** that take values *x1, x2, ……..,xn* with probabilities P_X(*x1*), P_X(*x2), ……,* P_X(*xn)* , respectively. But because Y= g(X), Y takes values g(*x1*), g(*x2*),……..g(*xn*) , with probabilities P_X(*x1*), P_X(*x2), ……,* P_X(*xn) r*espectively.

Hence $\bar{Y} = \overline{g(x)} = \sum_{i=1}^{n} g(xi)\ P_X(xi)$

If X is a continuous RV, then,

$\bar{Y} = \overline{g(x)} = = \int_{-\infty}^{\infty} g(x)\ f_X(x)\ dx$

Mean of the Sum:

If g1(X, Y), g2(X, Y),……,gn(X, Y) are functions of the RVs X and Y, then,

$\overline{g1(X,Y) + g2(X,Y) + \cdots \ldots \ldots + gn(X,Y)} = \overline{g1(X,Y)} + \overline{g2(X,Y)} + \cdots + \overline{gn(X,Y)}$

Thus the mean of sum equals to the sum of means.

An important special case is $\overline{X + Y} = \bar{X} + \bar{Y}$

Mean of Product of Two Functions:

Let g(X, Y) = g1(X) g2(Y), then,

$\overline{g(X,Y)} = \overline{g1(X)\ g2(Y)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g1(X)g2(Y)\ f_{XY}(x,y)dx\ dy$

If X and Y are independent , then

$p_{XY}(x,y) = f_X(x)\ f_Y(y)$

and$\overline{g(X,Y)} = \overline{g1(X)\ g2(Y)} = \int_{-\infty}^{\infty} g1(x)\ f_X(x)\ dx\ \int_{-\infty}^{\infty} g2(y)\ f_Y(y)dy = \overline{g1(X)}\overline{g2(Y)}$ if X and Y are independent.

A special case of this is- $\overline{XY} = \bar{X}\bar{Y}$ if X and Y are independent.

Moments:

The n [th]**moment** of an RV X is defined as the mean value of X^n. thus, the n [th] moment of an RV X is

$$\overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x)\ dx$$

For n=1 i.e 1[st]**moment** of an RV X is defined as
$\bar{X} = \int_{-\infty}^{\infty} x\ f_X(x)\ dx$ is same as the mean value of RV X.

For n=2 i.e 2[nd]**moment** of an RV X is defined as
$\overline{X^2} = \int_{-\infty}^{\infty} x^2 f_X(x)\ dx$ is also known as the mean square (ms) value of RV X.

The n [th]**centralmoment** of an RV X is defined as

$\overline{(X-\bar{X})^n} = \int_{-\infty}^{\infty}(x-\bar{X})^n f_X(x)\ dx$

The 2[nd]**centralmoment** of an RV X is called the **variance** of X and is denoted by σ_x^2 , where σ_x is known as the **standard deviation (S.D.)** of the RV x.

By definition, $\sigma_x^2 = \overline{(X-\bar{X})^2} = \overline{X^2} - 2\overline{X}\overline{X} + \overline{X}^2 = \overline{X^2} - 2\overline{X}^2 + \overline{X}^2 = \overline{X^2} - \overline{X}^2$

Thus the variance of X is equal to the mean square value minus the square of the mean.

Some Useful Probability Density Functions:

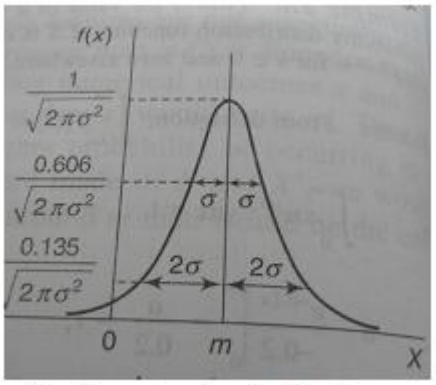
Some commonly used probability density functions are briefly discussed below:

1) **The Gaussian Probability Density:**The Gaussian (also called normal) probability density function is defined as

$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(x-m)^2}{2\sigma^2}}$ where m is the mean and σ^2 is the variance associated with f(*x*) .

We find that $m = \int_{-\infty}^{\infty} x\ \frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(x-m)^2}{2\sigma^2}}\ dx$ and $\sigma^2 = \int_{-\infty}^{\infty}(x-m)^2\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(x-m)^2}{2\sigma^2}}\ dx$

It may also be verified that$\int_{-\infty}^{\infty} x\ f(x)dx = 1$ as is required for a PDF.

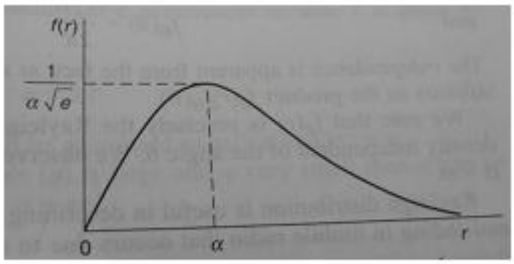


The Gaussian density function

As shown in figure, when x- m = $\pm\sigma$, i.e , at values of x separated from m by the standard deviation, f(x) has fallen to 0.606 of its peak value. When when$x$- m = $\pm 2\sigma$, f(x) falls to 0.135 of its peak value.

2) **The Rayleigh Probability Density:**Let the symbol represents an RV and the symbol r represent the value assumed by the variable. Then the Rayleigh PDF is defined as

$$f(r) = \begin{cases} \frac{r}{\alpha^2} e^{-r^2/2\alpha^2} & ,0 \le r \le \infty \\ 0 & ,r < 0 \end{cases} \quad \text{.....(1)}$$



The Rayleigh density

Here f(r) is non-zero only for positive value of r.a plot of f(r) as a function of r is shown in figure. It attains a maximum value 1/α $\sqrt{e}$at r= α. It may be verified that the mean value $\overline{R} = \sqrt{\frac{\pi}{2\,\alpha}}$, and the mean square value $\overline{R^2}$ = 2 α^2, and the variance σ_r^2= (2 – π/2) α^2.

3) **The Rician Distribution:**

The Rician distribution can be expressed as follows

$f_X(x) = \frac{x}{\sigma^2} e^{\frac{x^2+a^2}{2\sigma^2}} I_0\left(\frac{a\,x}{\sigma^2}\right)$

where, I_0 represents modified Bessel function of the first kind, zeroeth order, σ is the standard deviation of the underlaying Gaussian process and a is amplitude of the sinusoid or a^2 is sum of square of means of two independent Gaussian peocesses. The plot of $f_X(x)$ versus x looks like right shifted version of Rayleigh distribution, higher the value of a, more the shift. However, as a increases, it approaches Gaussian distribution while at a=0, it is same as Rayleigh distribution.

RANDOM PROCESSES

Definition and Examples:

A random process X(t) describes the mapping of a random experiment with sample space S onto an ensemble of sample functions X(t, λi). For each point in time t1, X(t1)describes a random variable.

Example: Rolling a Die

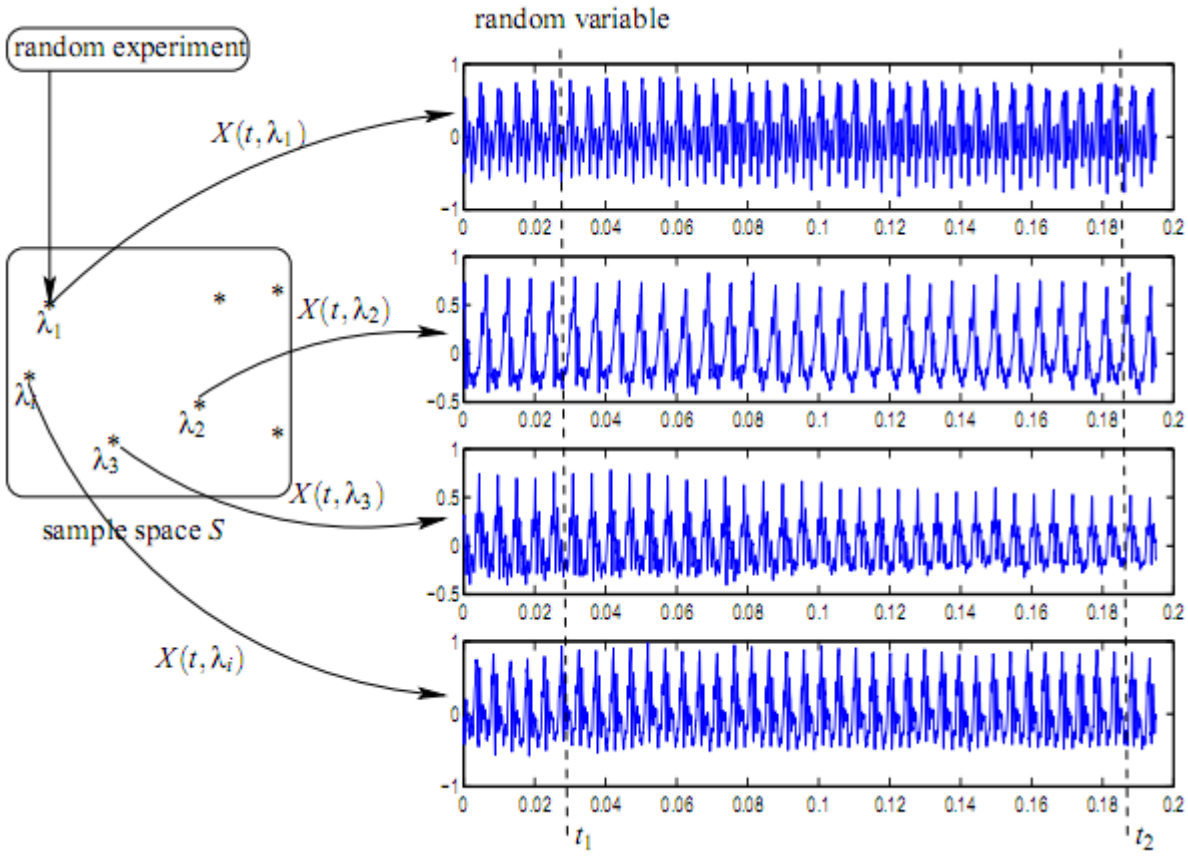
Random variable: X = i if number i is on top of the die

Random process: Y (t)= X · cos (ω0t).

Example: Tossing a Coin N Times

Random variable: Xn =0 if the n^{th} result is head, Xn =1 if the n t^hresult is tail

Randomprocess: $Y(t) = \sum_{n=1}^{N} X_n \cdot \text{rect}(t - n + 0.5).$



EnsembleAverages and Stationarity:

For each time instance of a random process, the average value, variance etc.can be Calculated from all sample functions X(t,λi).

Expected Value $E\{X(t)\}$**:**

For a random process $X(t)$ with probability density function $f_{X(t)}(x)$, the expected value $E\{X(t)\} = m_X(t)$ is given by:

$$E\{X(t)\} = \int_{-\infty}^{\infty} x \cdot f_{X(t)}(x)dx = m_X(t)$$

Variance $\sigma_X(t)$**:**

$$\sigma_X^2(t) = E\{|X(t) - m_X(t)|^2\} = \int_{-\infty}^{\infty} |x - m_X(t)|^2 \cdot f_{X(t)}(x)dx$$

For a **stationary** random process the probability density function is independent of time t, thus the expected value and the variance are also a constant over time.

$$f_{X(t)}(x) = f_{X(t+t_0)}(x), \qquad \forall \quad t, t_0$$

$$m_X(t) = m_X(t + t_0) = m_X, \qquad \sigma_X(t) = \sigma_X(t + t_0) = \sigma_X$$

TimeAveragesandErgodicity

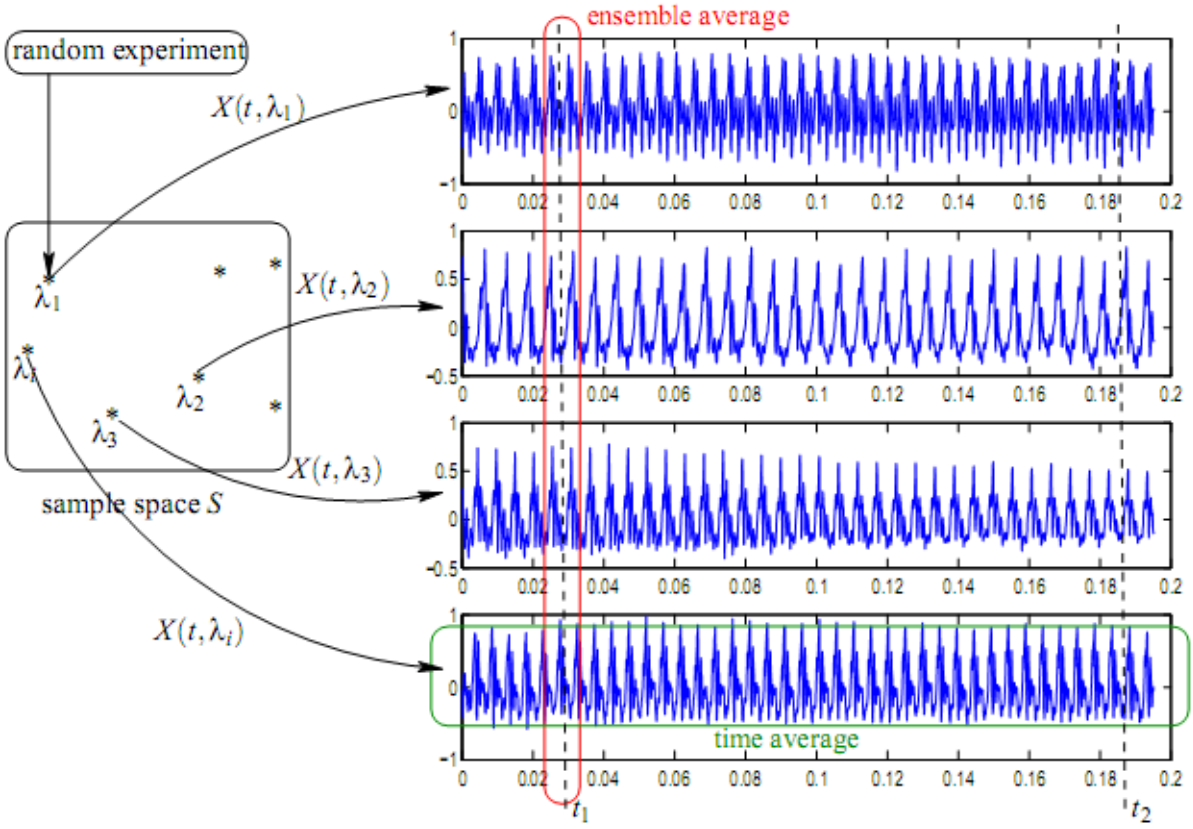
So far, the average value and the variance of a random process $X(t)$ were calculated based on the probability density function $f_{X(t)}$. However, in practical experiments the probability density function of a random process is often unknown. Also, in many cases, there is only one sample function $X(t, \lambda_i)$ available. Therefore, it is favorable to average over time instead of taking the ensemble average.

Average Value $m_{X(\lambda_i)}$**:**

$$m_{X(\lambda_i)} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t, \lambda_i) dt$$

Variance $\sigma^2_{X(\lambda_i)}$**:**

$$\sigma^2_{X(\lambda_i)} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} (X(t, \lambda_i) - m_{X(\lambda_i)})^2 dt$$



Ergodicity:

A stationary random process $X(t)$ is called **ergodic**, if the time averages of each sample function $X(t, \lambda_i)$ converge towards the corresponding ensemble average with probability one.

In practical applications ergodicity is often assumed since just one sample function is available and therefore the ensemble averages cannot be taken.

Example 1:

Random process: $X(t) = A\cos(\omega_0 t)$
A: discrete random variable with $P(A = 1) = P(A = 2) = 0.5$
ω_0: constant

Ensemble average:

$$m_X(t) = E\{X(t)\} = E\{A\}\cos(\omega_0 t) = 1.5\cos(\omega_0 t)$$

For $\omega_0 \neq 0$ the random process is not stationary and we are not allowed to take the time average.

Example 2:

Random process: $X(t) = A$
A: discrete random variable with $P(A = 1) = P(A = 2) = 0.5$

Ensemble average:

$$m_X(t) = E\{X(t)\} = E\{A\} = 1.5$$

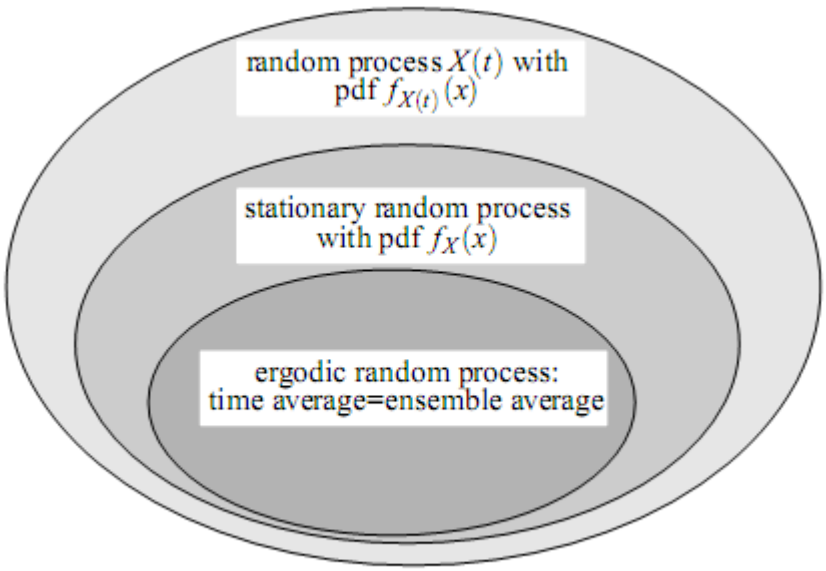
$\Rightarrow$ the ensemble average is independent of time.

Time averages:

$$m_{X(1)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t,1)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} 1\, dt = 1$$

$$m_{X(2)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t,2)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} 2\, dt = 2$$

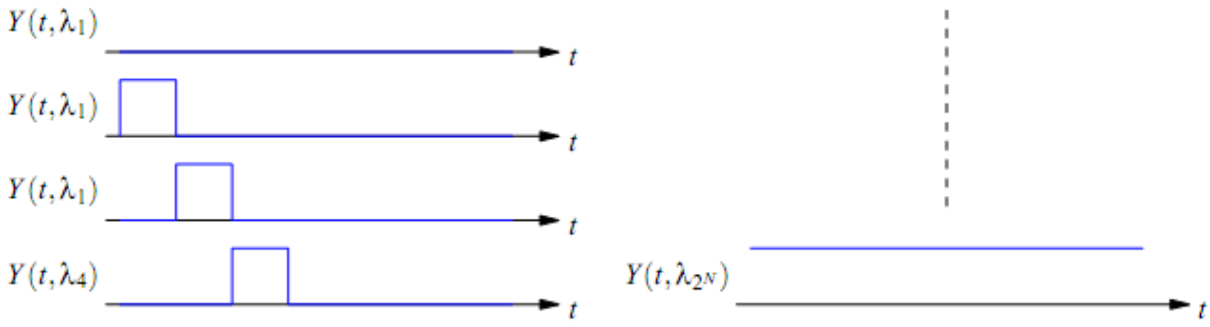
$\Rightarrow$ time averages taken for different sample functions are not identical to the ensemble average, the random process is thus not ergodic.



Example 3: Tossing a Coin N Times

Random variable: $X_n = 0$ if the nth result is head, $X_n = 1$ if the nth result is tail.

Random process: $Y(t) = \sum_{n=1}^{N} X_n \cdot \text{rect}(t - n + 0.5)$



Ensemble average:

$$m_Y(t) = E\{Y(t)\} = 0.5$$

Time average: Sample function $Y(t, \lambda_i)$: Coin is tossed N times and we observe n_1 times head and n_2 times tail.

$$m_{Y(\lambda_i)} = \frac{1}{N}\int_0^N Y(t,\lambda_i)dt = \frac{1}{N}(n_1 \cdot 0 + n_2 \cdot 1) = \frac{n_2}{N}$$

$\Rightarrow$ for $N \to \infty$, n_1 and n_2 converge towards $N/2$ and $m_{Y(t,\lambda_i)} = m_Y(t) = m_Y$. The random process is thus ergodic.

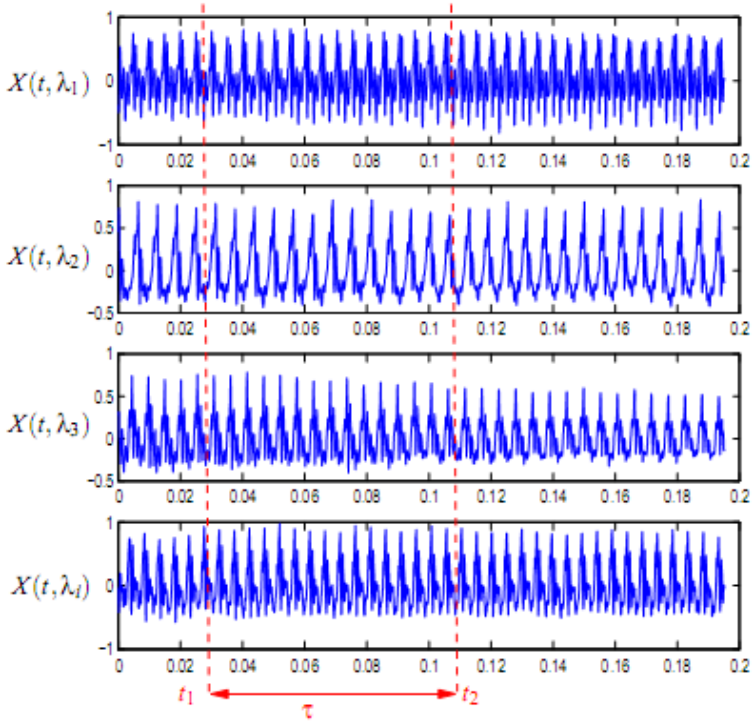
Autocorrelation and Autocovariance:

We are interested in how the value of a random process $X(t)$ evaluated at t_2 depends on its value at time t_1.

At t_1 and t_2 the random process is characterized by random variables X_1 and X_2, respectively.

The relationship between X_1 and X_2 is given by the joint probability density function

$$f_{X_1 X_2}(x_1, x_2)$$



Autocorrelation Function:

$$R_{XX}(t_1, t_2) = E\{X_1 X_2\}$$

$$= E\{X(t_1)\, X(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1 X_2}(x_1, x_2)dx_1 dx_2$$

Autocovariance Function:

$$C_{XX}(t_1, t_2) = E\{(X(t_1) - m_X(t_1))\,(X(t_2) - m_X(t_2))\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - m_X(t_1))(x_2 - m_X(t_2))\, f_{X(t_1)X(t_2)}(x_1, x_2)dx_1dx_2$$

$$= R_{XX}(t_1, t_2) - m_X(t_1)m_X(t_2)$$

$C_{XX}(t, t)$ describes the variance $\sigma_X^2(t)$ of a random process.

Autocorrelation and Autocovariance Function of a Stationary Random Process:

The joint probability density function of a stationary process does not change if a constant value t is added to both t_1 and t_2.

$$f_{X_1X_2}(x_1, x_2) = f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(t_1+t)X(t_2+t)}(x_1, x_2)$$

The autocorrelation function then only depends on the difference τ between t_1 and t_2

$$R_{XX}(t_1, t_2) = E\{X(t_1)\,X(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f_{X(t_1)X(t_2)}(x_1, x_2)dx_1dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f_{X(0)X(t_2-t_1)}(x_1, x_2)dx_1dx_2 = R_{X,X}(0, t_2 - t_1) = R_{X,X}(\tau)$$

Since the average value is a constant, the autocovariance function is given by:

$$C_{XX}(t_1, t_2) = E\{(X(t_1) - m_X)\,(X(t_2) - m_X)\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - m_X)(x_2 - m_X)\, f_{X(t_1)X(t_2)}(x_1, x_2)dx_1dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - m_X)(x_2 - m_X)\, f_{X(0)X(t_2-t_1)}(x_1, x_2)dx_1dx_2$$

$$= C_{XX}(0, t_2 - t_1) = C_{XX}(\tau)$$

Properties of the Autocorrelation Function of a Stationary Random Process:

- Symmetry: $R_{XX}(\tau) = R_{XX}(-\tau)$
- Mean Square Average: $R_{XX}(0) = E\{X(t)^2\} \geq 0$
- Maximum: $R_{XX}(0) \geq |R_{XX}(\tau)|$
- Periodicity: if $R_{XX}(0) = R_{XX}(t_0)$, then $R_{XX}(\tau)$ is periodic with period t_0.

Wide Sense Stationary (WSS) Random Process:

A random process $X(t)$ is called **WSS** if the following three properties are satisfied:

- The average value of the random process is a constant: $m_X(t) = m_X$
- The autocorrelation and autocovariance function only depend on the time difference $\tau = t_1 - t_2$:

$$R_{XX}(t_1, t_2) = R_{XX}(t_2 - t_1) = R_{XX}(\tau)$$

$$C_{XX}(t_1, t_2) = C_{XX}(t_2 - t_1) = C_{XX}(\tau)$$

- The variance is constant and finite: $\sigma_X^2 = C_{XX}(0) = R_{XX}(0) - m_X^2 < \infty$

Autocorrelation and Autocovariance Function of an Ergodic Random Process:

$$R_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X_T(t, \lambda_i) X_T(t+\tau, \lambda_i) dt$$

$$C_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} (X_T(t, \lambda_i) - m_X)(X_T(t+\tau, \lambda_i) - m_X) dt$$

$X_T(t, \lambda_i)$: sample function of random process $X(t)$ windowed to be of length T (starting at $-T/2$ ending at $T/2$).

Power Spectral Density:

Motivation:

- Description of random processes in the frequency domain
- Calculation of the Fourier Transform of a sample function is not useful
- We assume in the following that the random process considered is at least WSS if not stationary.

The power spectral density (psd) of a WSS random process $X(t)$ is defined as the Fourier Transform of the autocorrelation function $R_{XX}(\tau)$:

$$S_{XX}(f) = \mathcal{F}\{R_{XX}(\tau)\} = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j2\pi f\tau} d\tau$$

Inverse transform:

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f)\, e^{j2\pi f\tau}\, df$$

Properties of the Power Spectral Density:

$$S_{XX}(f) = S_{XX}(-f), \qquad S_{XX}(f) \geq 0, \qquad \mathrm{Im}\{S_{XX}(f)\} = 0$$

Ergodic Random Process $x(t)$:

Autocorrelation Function:

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_T(t,\lambda_i) x_T(t+\tau,\lambda_i) dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) x_T(t+\tau) dt$$

Power Spectral Density:

$$\begin{aligned} S_{xx}(f) &= \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j2\pi f\tau} d\tau \\ &= \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) x_T(t+\tau) dt\; e^{-j2\pi f\tau} d\tau \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) \underbrace{\int_{-\infty}^{\infty} x_T(t+\tau)\, e^{-j2\pi f\tau} d\tau}_{X_T(f) e^{j2\pi ft}}\, dt \\ &= X_T(f) \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_T(t)\, e^{j2\pi ft}\, dt \\ &= \lim_{T\to\infty} \frac{X_T(f) X_T^*(f)}{T} = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} \end{aligned}$$

Random Binary Wave:

A binary sequence is transmitted by rectangular pulses of width T_b. The amplitude of the pulse is determined by each bit, i.e. it is one if the bit is one and zero if the bit is zero. We assume that ones and zeros are equally likely and that each bit is statistically independent of all others. Using ergodicity, we obtain the following results from a sample function $x(t,1)$:

Average value (sample function of length N bits with n_1 ones):

$$m_x = E\{x(t,1)\} \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t,1) dt = \lim_{N\to\infty} \frac{1}{NT_b} \int_0^{NT_b} x(t,1) dt$$

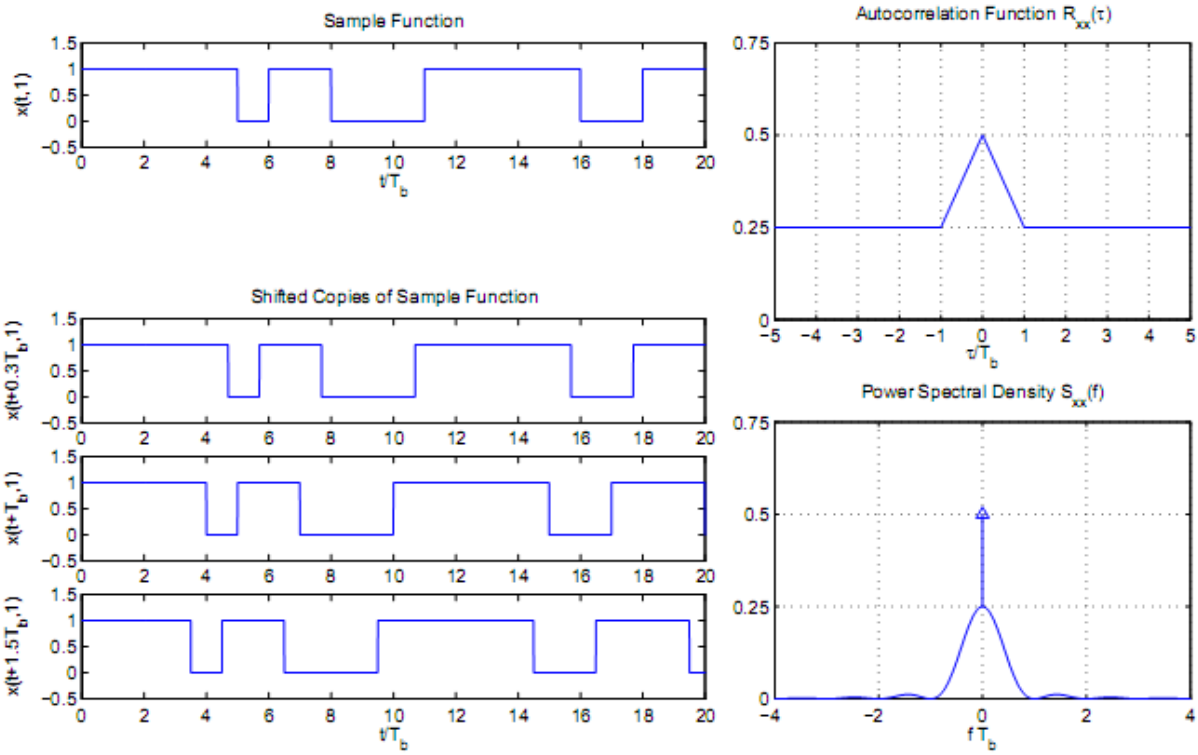
$$= \lim_{N\to\infty} \frac{1}{NT_b} [n_1 \cdot 1 \cdot T_b + (N-n_1) \cdot 0 \cdot T_b] = \lim_{N\to\infty} \frac{n_1}{N} = 0.5$$

Autocorrelation function:

$$\begin{aligned} R_{xx}(\tau) &= E\{x(t,1)x(t+\tau,1)\} = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t,1)x(t+\tau,1) dt \\ &= 0.25(\delta(t) + \Lambda(\tau/T_b)) \end{aligned}$$

Power spectral density:

$$S_{xx}(f) = \mathcal{F}\{R_{xx}(\tau)\} = 0.25(1 + \mathrm{sinc}^2(fT_b))$$


Sample Function
x(t,1)
t/T_b
Autocorrelation Function R_xx(τ)
τ/T_b
Shifted Copies of Sample Function
x(t+0.3T_b,1)
x(t+T_b,1)
x(t+1.5T_b,1)
t/T_b
Power Spectral Density S_xx(f)
f T_b

Error Vector Magnitude (**EVM**)

The **error vector magnitude** or **EVM** (sometimes also called **receive constellation error** or **RCE**) is a measure used to quantify the performance of a digital radio transmitter or receiver. A signal sent by an ideal transmitter or received by a receiver would have all constellation points precisely at the ideal locations, however various imperfections in the implementation (such as carrier leakage, low image rejection ratio, phase noise etc.) cause the actual constellation points to deviate from the ideal locations. Informally, EVM is a measure of how far the points are from the ideal locations.

Noise, distortion, spurious signals, and phase noise all degrade EVM, and therefore EVM provides a comprehensive measure of the quality of the radio receiver or transmitter for use in digital communications.Transmitter EVM can be measured by specialized equipment, which demodulates the received signal in a similar way to how a real radio demodulator does it. One of the stages in a typical phase-shift keying demodulation process produces a stream of I-Q points which can be used as a reasonably reliable estimate for the ideal transmitted signal in EVM calculation.

An **error vector** is a vector in the I-Q plane between the ideal constellation point and the point received by the receiver. In other words, it is the difference between actual received symbols and ideal symbols. The average power of the error vector, normalized to signal power, is the EVM. For the percentage format, root mean square (RMS) average is used.

The error vector magnitude is equal to the ratio of the power of the error vector to the root mean square (RMS) power of the reference. It is defined in dB as:

$$\mathrm{EVM(dB)} = 10\log_{10}\left(\frac{P_{\mathrm{error}}}{P_{\mathrm{reference}}}\right)$$

whereP_{error} is the RMS power of the error vector. For single carrier modulations, $P_{reference}$ is, by convention, the power of the outermost (highest power) point in the reference signal constellation. More recently, for multi-carrier modulations, $P_{reference}$ is defined as the reference constellation average power.[2]

EVM is defined as a percentage in a compatible way:

$$\mathrm{EVM(\%)} = \sqrt{\frac{P_{\mathrm{error}}}{P_{\mathrm{reference}}}} * 100\%$$

with the same definitions.

EVM, as conventionally defined for single carrier modulations, is a ratio of a mean power to a peak power. Because the relationship between the peak and mean signal power is dependent on constellation geometry, different constellation types (e.g. 16-QAM and 64-QAM), subject to the same mean level of interference, will report different EVM values.

EVM, as defined for multi carrier modulations, is arguably the more satisfactory measurement because it is a ratio of two mean powers and is insensitive to the constellation geometry. In this form, EVM is closely related to Modulation error ratio, the ratio of mean signal power to mean error power.

Vector Signal Analyzer (VSA)

A **vector signal analyzer** is an instrument that measures the magnitude and phase of the input signal at a single frequency within the IF bandwidth of the instrument. The primary use is to make in-channel measurements, such as error vector magnitude, code domain power, and spectral flatness, on known signals.

Vector signal analyzers are useful in measuring and demodulating digitally modulated signals like W-CDMA, LTE, andWLAN. These measurements are used to determine the quality of modulation and can be used for design validation and compliance testing of electronic devices.

The vector signal analyzer spectrum analysis process typically has a down-convert & digitizing stage and a DSP & display stage.

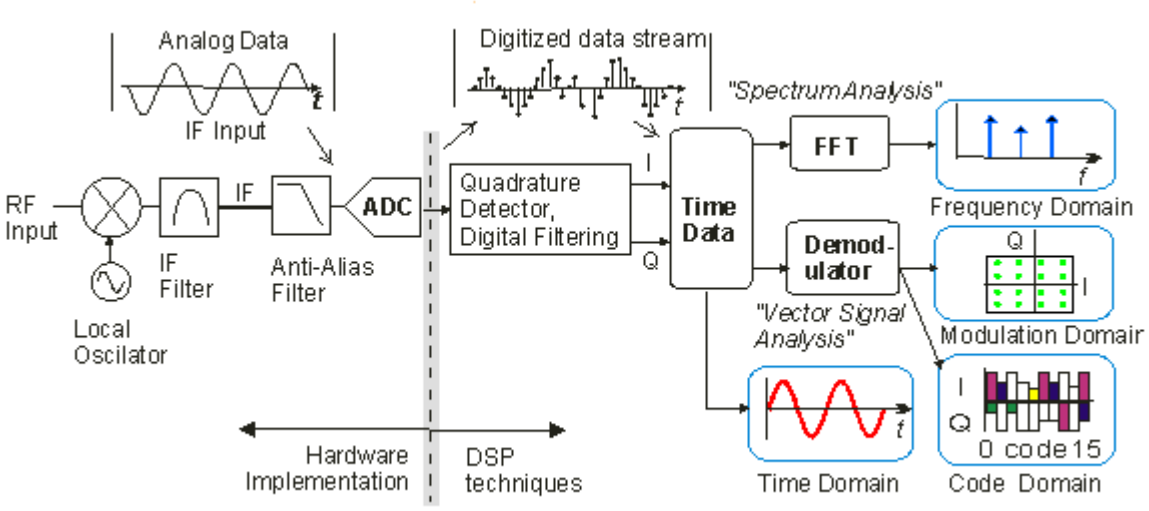
Down Convert & Digitize Stage

A vector signal analyzer operates by first down converting the signal spectra by using superheterodyne techniques. A portion of the input signal spectrum is down-converted (using a voltage-controlled oscillator and a mixer) to the center frequency of a band-pass filter. The use of a voltage-controlled oscillator allows for consideration of different carrier frequencies.

After the conversion to an intermediate frequency, the signal is filtered in order to band-limit the signal and preventaliasing. The signal is then digitized using an analog-to-digital converter. Sampling rate is often varied in relation to the frequency span under consideration.

DSP & Display stage

Once the signal is digitized, it is separated into quadrature and in-phase components using a quadrature detector, which is typically implemented with a discrete Hilbert transform. Several measurements are made and displayed using these signal components and various DSP processes, such as the ones below



Signal Spectrum from FFT

A FFT is used to compute the frequency spectrum of the signal. Usually there is a windowing function option to limit spectral leakage and enhance frequency resolution.[2] This window is implemented by multiplying it with the digitized values of the sample period before computing the FFT.